

# NOVEL INSTABILITY IN SUPERSTRING COSMOLOGY

JIRO SODA and MASA-AKI SAKAGAMI

*Department of Fundamental Sciences, FIHS, Kyoto University*

*Kyoto 606-8501, Japan*

*E-mail: jiro@phys.h.kyoto-u.ac.jp*

*E-mail: sakagami@phys.h.kyoto-u.ac.jp*

SHINSUKE KAWAI

*Graduate School of Human and Environmental Studies*

*Kyoto University, Kyoto 606-8501, Japan*

*E-mail: kawai@phys.h.kyoto-u.ac.jp*

As a most promising candidate for quantum theory of the gravity, the superstring theory has attracted many researchers including cosmologists. It is expected that the cosmological initial singularity is avoided within the context of the superstring theory. Indeed, Antoniadis et. al. found an interesting example of the non-singular cosmological solutions by considering the string one-loop correction. Here, we will discuss the stability of this model and find a new kind of instability in the graviton modes. We also argue this instability will persist even in the non-perturbative regime. The instability we have found might provide a mechanism to produce the primordial black holes.

## 1 Introduction

As is well known, the singularity is unavoidable consequence of the general relativity. The notorious cosmological singularity is nothing but the manifestation of this general theorem. Usually, we expect that the quantum gravity works at the Planck scale and the singularity disappear. To date, the superstring theory is the most promising candidate for the consistent theory of quantum gravity. Hence, it is interesting to seek the non-singular cosmological solutions in the low energy effective action of the superstring theory. From this point of view, Antoniadis et. al.<sup>1</sup> studied the one-loop corrected heterotic superstring effective action and found a particular interesting class of solutions which avoid the initial singularity.

The purpose of this work is to show that these solutions are unstable due to the high frequency gravitational wave. It means that the background geometry is not viable in the most cases. When we allow the fine tuning, however, the instability leads to an interesting consequence. If, at the arbitrary initial time, the amplitude is sufficiently small, then the small scale high amplitude gravitational waves will be produced by the instability without destroying the background geometry. In the deceleration phase, they will collapse and become

the primordial black holes. To support the above picture, we will discuss the non-perturbative effects phenomenologically and demonstrate that the instability will persist even in the non-perturbative regime.

## 2 Non-singular Cosmological Solution

From the non-linear sigma model approach, we can calculate the  $\beta$  function of the coupling function. The Weyl invariance requires the vanishing of the  $\beta$  function, which gives the equations of motion for the massless modes. From the equations of motion, the effective action can be easily read off. When we fix the world surface as the sphere, we obtain the tree level effective action. In actual calculation, we usually expand the effective action with respect to the string tension which controls the field theoretical loop expansion. The string loop correction is different from this. The string loop expansion corresponds to the path-integral quantization on the higher genus manifold. In particular, the one-loop string correction can be obtained from the calculation of the  $\beta$  function on the torus. It is this kind of correction that we want to investigate in detail.

The action for the heterotic superstring with orbifold compactification is given by<sup>1</sup>

$$S = \int \sqrt{-g} \left\{ \frac{1}{2} R - \frac{1}{4} (D\Phi)^2 - \frac{3}{4} (D\sigma)^2 + \frac{1}{16} [\lambda_1 e^\Phi - \lambda_2 \xi(\sigma)] R_{GB}^2 \right\} \quad (1)$$

where  $R, \Phi$  and  $\sigma$  are the scalar curvature, dilaton, and the modulus field, respectively. The Gauss-Bonnet curvature is given by

$$R_{GB}^2 = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \quad (2)$$

and the coupling function can be written by using the Dedekind  $\eta$  function:

$$\xi(\sigma) = -\log[2e^\sigma \eta(ie^\sigma)] = \xi(-\sigma) \quad (3)$$

For the existence of the non-singular cosmological solutions, the modulus field  $\sigma$  is relevant. Therefore, from now on, we will ignore the dilaton part of the action. Thus, we shall examine the modulus part of the effective action<sup>1,2</sup>

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} R - \frac{1}{4} (D\phi)^2 - \frac{1}{16} \lambda \xi(\phi) R_{GB}^2 \right\} . \quad (4)$$

Under the homogeneous and isotropic ansatz

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j , \quad (5)$$

we obtain the equations of motion:

$$-3H^2 = -\frac{1}{2}\dot{\phi}^2 - \frac{3\lambda}{2}H^3\dot{\xi} \quad (6)$$

$$-[2\dot{H} + 3H^2] = \frac{1}{2}\dot{\phi}^2 - \frac{\lambda}{2}[H^2\ddot{\xi} + 2H\dot{\xi}(\dot{H} + H^2)] , \quad (7)$$

where  $H = \dot{a}/a = da/dt/a$ .

It is convenient to define the effective energy density and pressure as  $\rho_{\text{eff}} = -G_0^0$  and  $p_{\text{eff}} = G_0^0/3$ . Then,

$$\Gamma = \frac{\rho_{\text{eff}} + p_{\text{eff}}}{\rho_{\text{eff}}} = -\frac{2\dot{H}}{3H^2} . \quad (8)$$

Here, we would like to stress that this effective adiabatic index is that of the background effective matter. It is not related to the adiabatic index of the perturbative matter.

Utilizing the asymptotic form

$$\xi_{,\phi} \sim \text{sign}(\phi) \frac{\pi}{3} e^{|\phi|} , \quad (9)$$

it is possible to obtain the asymptotic solutions by imposing the ansatz

$$H = \omega_1 |t|^\beta \quad (10)$$

$$\phi = \phi_0 + \omega_2 \log |t| . \quad (11)$$

In the asymptotic future, we expect the Friedman-Robertson-Walker universe, hence assume that the Einstein part balances with the ordinary energy momentum in the future. The asymptotic solution in  $t \rightarrow \infty$  becomes

$$a \sim t^{\frac{1}{3}} , \quad (12)$$

$$H = \frac{1}{3} \frac{1}{t} . \quad (13)$$

This is the Friedman-Robertson-Walker solution with stiff matter. For the later purpose, we need to calculate the effective adiabatic index. From the asymptotic solutions, it is easy to calculate  $\Gamma$  as

$$\Gamma = 2 , t \rightarrow \infty . \quad (14)$$

In the asymptotic past,  $t \rightarrow -\infty$ , in order to obtain the non-singular solution, the Gauss-Bonnet part must balance with the ordinary energy momentum. Then, we obtain the super-inflationary phase

$$H \sim \frac{1}{(-t)^2} . \quad (15)$$

This type of solutions appears in the pre-big-bang scenario. The serious problem there is the graceful exit problem. Interestingly, one-loop corrected action gives a solution to this problem. The effective adiabatic index becomes

$$\Gamma = \frac{4t}{3\omega_1}, t \rightarrow -\infty. \quad (16)$$

Numerical calculation tells us that the above asymptotic solutions are smoothly connected. This is an example of the graceful exit in the pre-big-bang scenario.<sup>3</sup>

Apparently, in the Gauss-Bonnet dominated phase, the energy condition is violated strongly. This fact itself is expected generally to obtain the non-singular cosmological solutions.

### 3 Stability analysis

Intuitively, it is expected to have the instability in the background which breaks the energy condition. In case of the perfect fluid, the negative pressure causes the instability and the inhomogeneity is enhanced catastrophically. However, what kind of instability can we imagine in this field theoretical context? Does it happen in the scalar perturbation, vector perturbation, or tensor perturbation? Unfortunately, we do not have any answer without calculation. Indeed, we performed numerical calculations and found no essential instability in the scalar and vector perturbation. It is the tensor perturbation mode that shows the instability. Here is something interesting. The isotropic and homogeneous background drives the instability of the gravitational wave modes. This counter intuitive result prevents us to use the fluid analogy. After all, the analogy is nothing but analogy. Another interesting feature of this instability is its scale dependence. On the contrary to the Jeans instability, the instability becomes strong in the high frequency graviton modes. Hence, the time scale of the instability could be arbitrarily small as far as the cut off scale is not introduced.

To illustrate the instability explicitly, it is convenient to see the Hamiltonian of the system. Consider the tensor perturbation

$$ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dx^j \quad (17)$$

where  $h^i_j = h^i_i = 0$ . By using the formula listed in the Appendix, we obtain the action for the tensor perturbation as

$$S = \frac{1}{8} \int d^4x a^3 [\dot{h}_{ij} \dot{h}^{ij} - \frac{1}{a^2} \nabla h_{ij} \nabla h^{ij} - (4\dot{H} + 6H^2 + \dot{\phi}^2) h_{ij} h^{ij}] \\ - \frac{\lambda}{16} \int d^4x a^3 \{ \ddot{\xi} [-a \nabla h_{ij} \nabla h^{ij} - 2a^3 H^2 \dot{h}_{ij} \dot{h}^{ij}]$$

$$\begin{aligned}
& -\dot{\xi}[-a^3 H \dot{h}_{ij} \dot{h}^{ij} + 4a^3 (\dot{H} + H^2) H h_{ij} h^{ij}] \} \\
& = \frac{1}{8} \int d^4 x a^3 \left[ \left(1 - \frac{\lambda}{2} H \dot{\xi}\right) \dot{h}_{ij} \dot{h}^{ij} - \left(1 - \frac{\lambda}{2} \ddot{\xi}\right) \frac{1}{a^2} \nabla h_{ij} \nabla h^{ij} \right]
\end{aligned}
\tag{18}$$

$$\tag{19}$$

The last expression is obtained by using the background equations. From the background equation, we also get

$$\begin{aligned}
1 - \frac{\lambda}{2} \ddot{\xi} &= \left(1 - \frac{\lambda}{2} H \dot{\xi}\right) \left(-\frac{2\dot{H}}{H^2} - 5\right) \\
&= \left(1 - \frac{\lambda}{2} H \dot{\xi}\right) (3\Gamma - 5)
\end{aligned}
\tag{20}$$

Notice that  $\alpha \equiv 1 - \frac{\lambda}{2} H \dot{\xi} > 0$ . Finally, we have obtained the action for the graviton modes:

$$S = \frac{1}{8} \int d^4 x a^3 \alpha \left[ \dot{h}_{ij} \dot{h}^{ij} - (3\Gamma - 5) \frac{1}{a^2} \nabla h_{ij} \nabla h^{ij} \right]
\tag{21}$$

It is straightforward to deduce the Hamiltonian of the system in the following form:

$$H = \int d^4 x \left[ \frac{2\pi^{ij}\pi_{ij}}{a^3\alpha} + (3\Gamma - 5) a \alpha \nabla h_{ij} \nabla h^{ij} \right],
\tag{22}$$

where  $\pi^{ij} = a^3 \alpha \dot{h}^{ij}/4$ .

If the energy condition is violated, i.e.  $\Gamma < 0$  or  $\Gamma < \frac{2}{3}$ , the system is unstable. Instability becomes strong as the wavelength becomes short.

In case of one-loop model, we have  $\Gamma = 2$  in the Friedmann-Robertson-Walker phase, so the background geometry is stable as is expected. On the other hand, it turns out that the Gauss-Bonnet phase is unstable due to the strong violation of the energy condition,  $\Gamma \sim -|t|$ . Strictly speaking, this instability immediately does not imply the breakdown of the background geometry. The point is that the fine tuning of the graviton amplitude is required to avoid the breakdown of the background geometry. It is possible to give the criterion for the breakdown of the background geometry explicitly.<sup>4</sup>

## 4 Discussion

First of all, it should be noted that the appearance of the Gauss-Bonnet term is generic features of the superstring cosmology. In case of the one-loop correction, the function form of the coupling between the modulus and the Gauss-Bonnet term is universal. Hence, our result can be regarded as the general feature at this level. However, our result suggest that the non-perturbative

effects should be taken into account if superstring theory is viable for solving the singularity problem. Assuming the Damour-Polyakov type universality, we can obtain the non-perturbative effective action

$$S = \int d^4x \sqrt{-g} \{ R - (\partial\Phi)^2 + B(\Phi) R_{GB}^2 + \dots \} \quad (23)$$

where

$$B(\Phi) = e^\Phi + c_0 + c_1 e^{-\Phi} + c_2 e^{-2\Phi} + \dots \quad (24)$$

Here, the function  $B(\Phi)$  contains the correction from the string higher loops. The coefficients  $c_i$  should take the values so that the initial singularity will be avoided. The energy condition is necessarily violated in this case again. Now, we shall show, even in this case, the instability we have found will appear. More generally, we can take into account other massless fields in the model:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [ R - \sum_i (\nabla\phi_i)^2 - \frac{\lambda}{8} G(\phi_i) R_{GB}^2 ] \quad (25)$$

where  $G(\phi_i)$  is now a function of massless fields  $\phi_i$ . The action for the tensor perturbation again takes the following form

$$S_{\text{perturb}} = \frac{1}{8} \int d^4x a^3 \left( 1 - \frac{\lambda}{2} H \dot{G}(\phi_i) \right) [ \dot{h}_{ij} \dot{h}^{ij} - (3\Gamma - 5) \frac{1}{a^2} \nabla h_{ij} \nabla h^{ij} ] \quad (26)$$

This is essentially the same as the previous form. So, we have proved the existence of the instability.

Next, we will argue that the fine tuning problem is not avoidable in the non-singular cosmology in this model. Let us assume the non-singular cosmological solution. Then, we have the eternal past and the eternal future. Of course, there exists a period where the energy condition is violated in order to have the non-singular cosmological solution. However, it is not allowed to have the infinite period of violating the energy condition, otherwise we will encounter the fine tuning problem of the graviton amplitudes. This implies that the effective adiabatic index  $\Gamma$  is positive, more precisely  $\Gamma > 5/3$  in most of the time. From the definition  $\Gamma = -2\dot{H}/3H^2$ , we obtain

$$\frac{d}{dt} \left( \frac{1}{H(t)} \right) = \frac{3}{2} \Gamma(t) \quad (27)$$

As  $\Gamma(t)$  is almost always positive,  $1/H(t)$  is almost monotonic function, hence  $1/H(t)$  becomes zero at a certain time  $t_0$ . This means

$$\lim_{t \rightarrow t_0} H(t) = \infty \quad (28)$$

namely we have encountered the singularity. Thus, as far as the non-singularity and the avoidance of the fine tuning is assumed, we must have the singularity somewhere. This is the contradiction. Hence, if we want to have the non-singular cosmological solutions in this context, the fine tuning problem can not be avoidable. Hence, this kind of non-perturbative correction can not change the situation. It should be noted that the spatial curvature might change the arguments above.<sup>5</sup>

An interesting consequence of our result is that the instability is stronger in the small scale. Hence, the instability found by us can provide a mechanism to generate the primordial black holes. We will investigate this possibility in the future.

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### Appendix

Here, we will list the necessary formula to calculate the perturbative action:

$$\begin{aligned}
R_{0i}^{(0)0j} R_{0j}^{(2)0i} &= (\dot{H} + H^2) \left[ -\frac{1}{2} h^{ik} \ddot{h}_{ik} - \frac{1}{4} \dot{h}^{ik} \dot{h}_{ik} - H h^{ik} \dot{h}_{ik} \right] \\
R_{ij}^{(0)kl} R_{kl}^{(2)ij} &= 2H^2 \left[ -\frac{1}{4a^2} h^{ik,m} h_{ik,m} - \frac{1}{4} \dot{h}^{ik} \dot{h}_{ik} - 2H h^{ik} \dot{h}_{ik} \right] \\
R_{0i}^{(1)0j} R_{0j}^{(1)0i} &= \frac{1}{4} \ddot{h}^{ik} \ddot{h}_{ik} + H \dot{h}^{ik} \ddot{h}_{ik} + H^2 \dot{h}^{ik} \dot{h}_{ik} \\
R_{kl}^{(1)0j} R_{0j}^{(1)kl} &= -\frac{1}{2a^2} \dot{h}^{ik,m} \dot{h}_{ik,m} \\
R_{kl}^{(1)ij} R_{ij}^{(1)kl} &= \frac{1}{a^4} \nabla^2 h^{ik} \nabla^2 h_{ik} - \frac{2H}{a^2} \dot{h}^{ik} \nabla^2 h_{ik} + H^2 \dot{h}^{ik} \dot{h}_{ik} \\
R_0^{(0)0} R_0^{(2)0} &= 3(\dot{H} + H^2) \left[ -\frac{1}{4} \dot{h}^{ik} \dot{h}_{ik} - \frac{1}{2} h^{ik} \ddot{h}_{ik} - H h^{ik} \dot{h}_{ik} \right] \\
R_j^{(0)i} R_i^{(2)j} &= (\dot{H} + 3H^2) \left[ -\frac{1}{2} h^{ik} \ddot{h}_{ik} - \frac{1}{2} \dot{h}^{ik} \dot{h}_{ik} - 3H h^{ik} \dot{h}_{ik} - \frac{1}{4a^2} h_{jk,m} h^{jk,m} \right] \\
R_j^{(1)i} R_i^{(1)j} &= \frac{1}{4} \ddot{h}^{ik} \ddot{h}_{ik} + \frac{1}{4a^4} \nabla^2 h_{jk} \nabla^2 h^{jk} + \frac{9}{4} H^2 \dot{h}^{ik} \dot{h}_{ik} \\
&\quad - \frac{1}{2a^2} \ddot{h}_{jk} \nabla^2 h^{jk} + \frac{3}{2} H \ddot{h}^{ik} \dot{h}_{ik} - \frac{3H}{2a^2} \dot{h}_{jk} \nabla^2 h^{jk} \\
R^{(0)} R^{(2)} &= 6(\dot{H} + 2H^2) \left[ -\frac{3}{4} \dot{h}^{ik} \dot{h}_{ik} - h^{ik} \ddot{h}_{ik} - 4H h^{ik} \dot{h}_{ik} - \frac{1}{4a^2} h_{jk,m} h^{jk,m} \right].
\end{aligned}$$

Here, we have omitted the spatially total derivative terms, because they do not contribute the final answer.

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